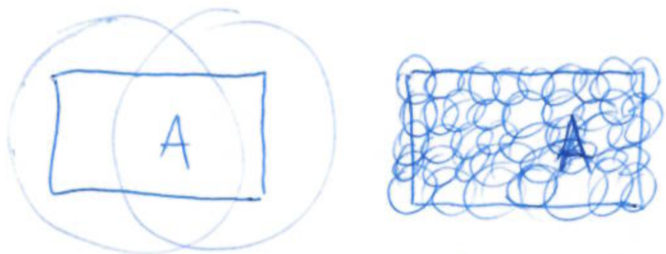
A few important notions from topology.

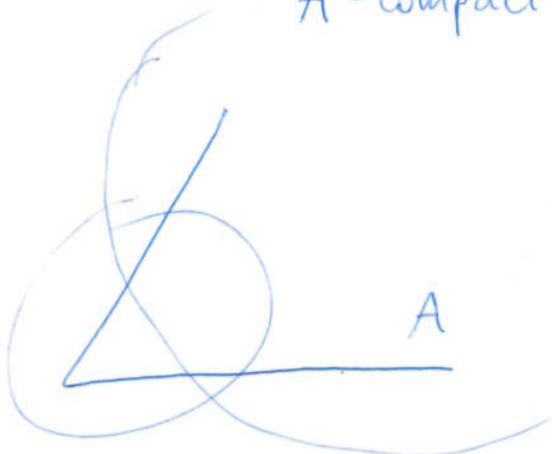
Def. Collection of sets $\mathcal{U} = \{U_i : i \in I\}$ in a metric space (X, d) is a <u>cover</u> of A if $A \subset \bigcup_{i \in I} U_i$. If $\{U_i\}$ are open, then it is an <u>open cover</u>.

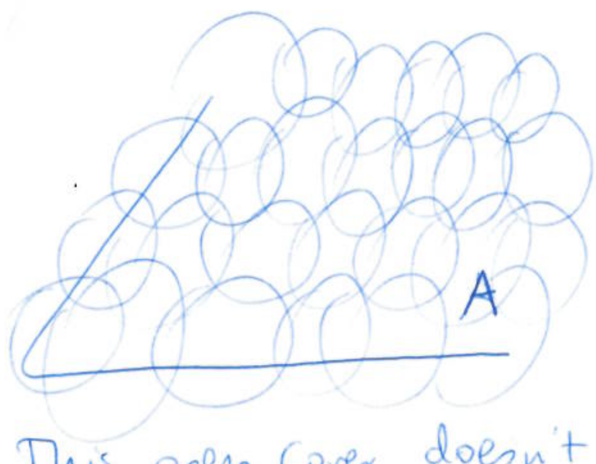
Def. (Compact set) Set A is <u>compact</u> if <u>every</u> open cover of A has a <u>finite</u> subcover $\{U_1, \dots, U_n\}$.

Ex. 1



A - compact





This open cover doesn't have a finite subcover.

A - not compact

Ex. 2

$(0,1) \subset \mathbb{R}$ is <u>not compact</u>. Let $\mathcal{U} = \{(\frac{1}{n}, 1)\}_{n \in \mathbb{N}}$.
This cover does not have a finite subcover.

Def. (Sequential compactness) Set A in a metric space (X,d) is sequentially compact if every sequence $\{x_n\}_{n\in\mathbb{N}} \subset A$ has a convergent subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$, $\lim_{k\to\infty} x_{n_k} \in A$.

A

A

Theorem A compact set in a metric space is sequentially compact.

Proof. (by contradiction)
Let A be not sequentially compact. Hence there exists a sequence $\{x_n\}$ without a convergent subsequence. For all $x \in A$ there exists $\varepsilon > 0$ such that $B_\varepsilon(x)$ has only a finite number of elements of $\{x_n\}$.
Obviously, $\mathbb{B} := \{B_\varepsilon(x)\}_{x\in A}$ is an open cover of A.
Compactness would imply that $\mathbb{B}$ has a finite subcover $\overline{\mathbb{B}}$. But then $\overline{\mathbb{B}}$ contains a finite number of elements of $\{x_n\}$. Hence $\{x_n\} \not\subset A$. Contradiction. $\square$

Theorem. A sequentially compact set in a metric space is compact.

Proof (Skipped)

Theorem. $B \subset X$, where B – closed, X – compact. Then B – compact.

Proof. Let $\mathcal{U} = \{U_i : i \in I\}$ be an open cover of X. Let B^c – complement ($B^c = X \setminus B$). Then B^c is open. Also, $\mathcal{U} \cup B^c$ is an open cover of X. As X – compact, this cover has a finite subcover $\{U_1, \dots, U_n\} \cup B^c$, and thus $\{U_1, \dots, U_n\}$ is a finite subcover of B. $\square$

Fact. $A \subset X$ – compact $\Rightarrow$ A – closed
$\Rightarrow$ A – bounded

~~Fact. $A \subset X$ – compact $\Leftrightarrow$~~

Theorem (Heine-Borel) In $\mathbb{R}^m$, it follows that
A – compact $\Leftrightarrow$ A – closed and bounded.

$\Rightarrow$: (Handbook)

$\Leftarrow$: Let $A \subset \mathbb{R}^m$ be closed and bounded. Then $\exists M > 0$ $\|x\| \leq M$, for all $x \in A$. Then $A \subset C_m$ where $C_m = \underbrace{I \times I \times \dots \times I}_{m \text{ times}}$, with $I = [-M, M]$ (hypercube). It is sufficient to show that C_m – compact.

Let $\{x_n\} \subset C_m$, with $x_n = (x_{1n}, x_{2n}, \ldots, x_{mn})$. 5

For all coordinates $i = 1, \ldots, m$, sequences of numbers $\{x_{in}\}_{n \in \mathbb{N}}$ have subsequences convergent to limits $\overline{x_i} \in I$ (by Bolzano-Weierstrass theorem, any closed interval in $\mathbb{R}$ is compact). Now we pick (dimension by dimension) $\{x_{1 n_{k_1}}\} \xrightarrow[k_1 \to \infty]{} \overline{x}_1$, $\{x_{2 n_{k_2}}\} \xrightarrow[k_2 \to \infty]{} \overline{x}_2$, $\ldots$, $\{x_{m n_{k_m}}\} \xrightarrow[k_m \to \infty]{} \overline{x}_m$, recursively. This subsequence converges to $(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_m) \in C_m$. Hence C_m is sequentially compact. $\square$

Theorem (Bolzano-Weierstrass)
Any bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Proof. Let $\{x_n\} \subset [a, b]$

Lemma: $\{x_n\}$ has a monotone subsequence.
Proof of lemma: Let n be a "peak" ($x_n > x_m$ for all $m > n$)
If there are infinitely many peaks, then $\{x_{n_1}, x_{n_2}, \ldots\}$ is a monotone subsequence. If there are finitely many peaks, then N – last peak. Then $n_1 = N+1$ is not a peak so there exists $n_2 > n_1$ with $x_{n_2} \geq x_{n_1}$. But then n_2 is not a peak, etc., and thus $x_{n_1} \leq x_{n_2} \leq \ldots$ is a monotone subsequence.

Hence there exists a convergent subsequence. $\square$
So $[a, b]$ is sequentially compact.

<u>Ex.</u> Prove, using the Borel-Lebesgue definition of compactness, that $[0,1]$ is compact.

Let $\mathcal{U} = \{U_i : i \in I\}$ be an open cover of $[0,1]$. Consider $A = \{x \in [0,1] : [0,x]$ is covered by finitely many sets $U_i\}$. Let α be the least upper bound of A. We know that $\alpha \in [0,1]$. Suppose $\underline{\alpha < 1}$. Then α is in U_{i_0}, which is an open set, so $\exists \varepsilon > 0$ $B_\varepsilon(\alpha) \subset U_{i_0}$. But now $[0, \alpha - \frac{\varepsilon}{2}]$ is also covered by finitely many U_i's, $\{U_1, \ldots, U_n\}$. And also $\{U_1, \ldots, U_n\} \cup U_{i_0}$ covers $[0, \alpha + \frac{\varepsilon}{2}]$ which contradicts the definition of α. So $\underline{\alpha = 1}$.

A8

Let $v = (v_1, v_2, \ldots)$.

Define $\|v\| := \sup_{t \geq 1} \{|v_t|\}$.

ⓐ • show that $\|\cdot\|$ is a well-defined norm.

ⓑ • show that if $\|v_{(n)} - v\| \xrightarrow{n\to\infty} 0$ then $\forall t \geq 1$, $|v_{(n)t} - v_t| \xrightarrow{n\to\infty} 0$

but the reverse implication does not hold.

Ad. a) • $\|v\| = 0 \iff \sup_{t\geq 1} |v_t| = 0 \iff \forall t \geq 1 \; v_t = 0 \iff v = 0$.

• $\|\alpha v\| = \sup_{t\geq 1} |\alpha v_t| = \sup_{t\geq 1} |\alpha| \cdot |v_t| = |\alpha| \sup_{t\geq 1} |v_t| = |\alpha| \cdot \|v\|$

• $\|v + w\| = \sup_{t\geq 1} |v_t + w_t| \leq \sup_{t\geq 1} \{|v_t| + |w_t|\} \leq$

$\leq \sup_{t\geq 1} |v_t| + \sup_{t\geq 1} |w_t| = \|v\| + \|w\|$

Ad. b) Let $\|v_{(n)} - v\| \xrightarrow{n\to\infty} 0$. Hence, $\sup_{t\geq 1} |v_{(n)t} - v_t| \xrightarrow{n\to\infty} 0$,

and so $\forall t \geq 1 \; |v_{(n)t} - v_t| \xrightarrow{n\to\infty} 0$. (Otherwise, if $\exists \varepsilon > 0 \; \forall \tilde{n} \; \exists n > \tilde{n}$

$|v_{(n)\bar{t}} - v_{\bar{t}}| \geq \varepsilon$, for some $\bar{t} \geq 1$, then $\sup_{t\geq 1} |v_{(n)t} - v_t| \geq \varepsilon$,

a contradiction.)

To show that the converse is not true, take a counterexample

E.g., ← "diagonal argument"

$v_{(1)} = (1, 0, 0, \ldots)$

$v_{(2)} = (0, 1, 0, \ldots)$

$v_{(3)} = (0, 0, 1, \ldots)$.

We see that: $\lim_{n\to\infty} v_{(n)t} = 0 \quad \forall t \geq 1$. Yet,

$\|v_{(n)} - 0\| = \sup_{t\geq 1} \{|v_{(n)t}|\} = 1 \quad \forall n \in \mathbb{N}$. So $\|v_{(n)} - 0\| \not\to 0$.

□

Example

$E = \{(u_n)_{n\in\mathbb{N}} : |u_n| \leq 1 \ \forall n \in \mathbb{N}\}$.

For all $u \in E$, let $\|u\| = \max\{|u_n|, n \geq 0\}$. Prove that there exists a seq. $(u(k)) \subset E$ without any convergent subsequence in $(E, \|\cdot\|)$, so E is not compact.

~~NORMS~~ NORMS OF VECTORS DON'T CONVERGE IN ANY SUBSEQUENCE

~~Diagonal approach:~~

~~$u(1) = \ldots$~~

Not true in $C^m = I \times I \times \ldots \times I$, $I = [-1, 1]$!

We would have that $\{u_1, u_2, \ldots\} \subset C^m$ – compact. Hence sequentially compact.

It turns out that $C^\infty = I \times I \times \ldots$ is not compact

(?)

Diagonal approach:

$$u(1) = (u_1^1, u_2^1, u_3^1, u_4^1, \ldots)$$
$$u(2) = (u_1^2, u_2^2, u_3^2, u_4^2, \ldots)$$
$$u(3) = (u_1^3, u_2^3, u_3^3, u_4^3, \ldots)$$
$$u(4) = (u_1^4, u_2^4, u_3^4, u_4^4, \ldots)$$
$$\vdots$$

⇒ Page 8

Theorem (Tychonoff). Product of any collection of compact topological spaces is compact (wrt to the induced product topology)

Corollary It holds true for metric spaces as well.

Warning: PRODUCT METRIC: $(X, d_1), (Y, d_2) \Rightarrow (X \times Y, d_\pi)$ with $d_\pi(z, z') = \sqrt{(d_1(x,x'))^2 + (d_2(y,y'))^2}$

Solution)

$d(u(i),u(j)) = \|u(i)-u(j)\| = \max\limits_{k\in\mathbb{N}} |u_{ik}-u_{jk}|$ $\boxed{!}$

Let $u(i) = (0,\dots,0,\underset{\substack{\uparrow\\ i\text{-th place}}}{1},0,\dots) \in E$.

We have $\|u(i)-u(j)\| = 1$ $\forall i,j\in\mathbb{N}$. $(u(k))_{k\in\mathbb{N}}$ does not have a convergent subsequence.

Hence, the space $\ell_\infty(E[0,1])$ is <u>not</u> compact.

(Even though there is convergence coordinate-wise.)

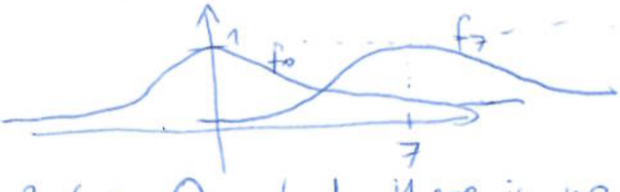
<u>Analogous example</u>

- Space $B[0,1]$ — space of bounded, continuous fcts $f: \mathbb{R}\to[0,1]$

The "sup" norm: $\|f\| = \sup\limits_{x\in\mathbb{R}} |f(x)|$. [UNIFORM CONVERGENCE]

Hence $d(f_i,f_j) = \|f_i-f_j\| = \sup\limits_{x\in\mathbb{R}} |f_i(x)-f_j(x)|$.

Let, e.g. $f_i(x) = e^{-(x-i)^2}$



$\forall x\in\mathbb{R}$ $\lim\limits_{n\to\infty} f_n(x) = 0$, <u>but</u> there is no uniform convergence as $\|f_n - 0\| = 1$ for all n.

~~$g(x) = f_i(x)-f_j(x)$ $g'(x) = -2(x-i)e^{-(x-i)^2}$~~

If there exists a limit, it must be $f(x)\equiv 0$, but it turns out there's none. $\square$

Hence, $B[0,1]$ — <u>not</u> compact, ~~even though~~

TUTORIAL 17, EXERCISE 3

Example

$$E = \{(u_n)_{n\in\mathbb{N}} : u_n \in \mathbb{R}\}$$

$$d(u,v) = \sum_{n\geq 0} \frac{\min\{1, |u_n - v_n|\}}{2^n}.$$

a) Well-defined distance (metric) fct.

1° $\forall (u, v \in E)$ $d(u,v) \geq 0$ and $d(u,v) \leq \sum_{n\geq 0} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2$

2° Axioms: $d(u,u) = \sum_{n\geq 0} \frac{\min\{1,0\}}{2^n} = 0$.

$$d(u,v) = \sum_{n\geq 0} \frac{\min\{1, |u_n - v_n|\}}{2^n} = d(v,u)$$

$$d(u,v) + d(v,z) = \sum_{n\geq 0} \left[\frac{\min\{1, |u_n - v_n|\} + \min\{1, |v_n - z_n|\}}{2^n}\right] =$$

$$\geq \sum_{n\geq 0} \left[\frac{\min\{2, 1 + |u_n - v_n|, 1 + |v_n - z_n|, |u_n - z_n|\}}{2^n}\right] \geq d(u,z).$$

"=" only if this case appears $\forall n \in \mathbb{N}$

b) $(u(k))$ — sequence of ~~sequences~~ sequences $u \in E$

$(u(k)) \to u \iff (u_n(k))_{k\in\mathbb{N}} \to u_n$ for all n.

$$\boxed{\forall_{\varepsilon>0} \exists_{\bar{k}} \forall_{k>\bar{k}} \left(d(u(k), u) < \varepsilon\right) \iff \sum_{n\geq 0} \frac{\min\{1, |u_n(k) - u_n|\}}{2^n} < \varepsilon \iff \sum_{n\geq 0} \frac{|u_n(k) - u_n|}{2^n} < \varepsilon}$$

if valid!

$\Leftarrow$: $u_n(k) \xrightarrow[k\to\infty]{} u_n$ $\forall n$ then $\forall_{\varepsilon>0} \exists_{\tilde{k}} \forall_{k>\tilde{k}} |u_n(k) - u_n| < \frac{\varepsilon}{2}$

and so for $k > \max\{\bar{k}, \tilde{k}\}$, $\sum_{n\geq 0} \frac{|u_n(k) - u_n|}{2^n} < \frac{\varepsilon}{2} \sum \left(\frac{1}{2}\right)^n = \varepsilon$.

$\Rightarrow$: $\sum_{n\geq N} \frac{|u_n(k) - u_n|}{2^n} < \varepsilon$ $\Rightarrow$ ~~$\sum \frac{|u_n(k) - u_n|}{2^n} \leq \frac{|u_n(k) - u_n|}{2^n} < \varepsilon \Rightarrow |$~~

for $k > \bar{k}$

"ERRATUM" (P. 9) continued

9A

Prove that $(u(k)) \subset E$ converges to $u \in E$, where (E, d), iff $(u_n(k))_{k\in\mathbb{N}} \xrightarrow[k\to\infty]{} u_n$ $\forall n\in\mathbb{N}$.

$\Longrightarrow$: Assume that $\sum_{n\geq 0} \frac{\min\{1, |u_n-v_n|\}}{2^n} = d(u,v)$.

We have
$$\forall \varepsilon>0\ \exists \tilde{k}\ \forall k>\tilde{k}\ \sum_{n\geq 0}\frac{\min\{1,|u_n(k)-u_n|\}}{2^n}<\varepsilon.$$

(By contradiction.) Assume furthermore that there exists $\bar{n}\in\mathbb{N}$ such that $\lim_{k\to\infty}|u_{\bar{n}}(k)-u_{\bar{n}}|>0$. Call the distance $\delta>0$.

Then $\sum_{n\geq 0}\frac{\min\{1, |u_n(k)-u_n|\}}{2^n} \geq \frac{\delta}{2^{\bar{n}}}$. Hence for $\varepsilon<\frac{\delta}{2^{\bar{n}}\cdot 2}$, for all $\tilde{k}\in\mathbb{N}$ there exist $k>\tilde{k}$ with $d(u(k),u)>\varepsilon$.
A contradiction. □

$\Longleftarrow$: Assume that $\forall n\in\mathbb{N}$, it is true that
$$\forall\varepsilon>0\ \exists \tilde{k}_{n,\varepsilon}\ \forall k>\tilde{k}_{n,\varepsilon}\ |u_n(k)-u_n|<\frac{\varepsilon}{4}.$$

However, the convergence is not uniform across n. The conclusion is only true because we have the $\min\{1,|\cdot|\}$ part, and thus we only have to care about the initial finite number of coordinates. We have:
$$\forall\varepsilon>0\ \exists\bar{n}\ \forall n>\bar{n}\ \sum_{m=n}^{+\infty}\frac{\min\{1,|u_m(k)-v_m|\}}{2^m}\leq\sum_{m=n}^{\infty}\left(\frac{1}{2}\right)^m=\left(\frac{1}{2}\right)^{n-1}\leq\left(\frac{1}{2}\right)^{\bar{n}-1}$$

Hence, for any $\varepsilon>0$ we pick a sufficiently large cutoff point $\bar{n}$, such that $\left(\frac{1}{2}\right)^{\bar{n}-1}<\frac{\varepsilon}{2}$. This is always possible. Then, we have:
$$\forall\varepsilon>0\ \exists\tilde{k}\ \forall k>\tilde{k}\ d(u(k),u)\leq\sum_{m=0}^{\bar{n}}\frac{\min\{1,|u_m(k)-v_m|\}}{2^m}+\left(\frac{1}{2}\right)^{\bar{n}-1}\leq$$
$$\leq\sum_{m=0}^{\bar{n}}\frac{|u_m(k)-v_m|}{2^m}+\frac{\varepsilon}{2}\leq\max_{m=1,\dots,\bar{n}}|u_m(k)-v_m|\cdot 2+\frac{\varepsilon}{2}<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$ □

FINITE!!

c) Prove that (E, d) is compact. 9 B (10)

Let us prove that it is sequentially compact.

We know that $u(k) \xrightarrow[k\to\infty]{} u \iff \forall_{n\in\mathbb{N}}\ u_n(k) \xrightarrow[k\to\infty]{} u_n$.

→ Since for all $n\in\mathbb{N}$ $(u_n(k))\subset\mathbb{R}$, by Bolzano-Weierstrass theorem we know that every $(u_n(k))$ has a convergent subsequence.

→ Take a sequence of sequences (infinite-dimensional vectors): $(u(k)) = (u_1(k), u_2(k), u_3(k), \dots)$. We have to show that $(u(k))$ has a convergent subsequence.

→ Let $u_1(k_l)$ be convergent to u_1 (as $l\to\infty$).

Let $u_2(k_{l_m})$ — " — u_2 (as $m\to\infty$)

Let $u_3(k_{l_{m_p}})$ — " — u_3 (as $p\to\infty$)

$\vdots$

Let $u_{\bar n}(k_{l_{m_{p\dots q}}})$ — " — $u_{\bar n}$ (as $q\to\infty$). [Stop at a finite $\bar n$.]

→ We have (for $u=(u_1, u_2, u_3, \dots)$), for large $q>\tilde q$,

$$d\left(u(k_{l_{m_{p\dots q}}})-u\right) \le \sum_{n=0}^{\bar n}\frac{|u_n(k_{l\dots q})-u_n|}{2^n} + \sum_{n=\bar n}^{+\infty}\left(\frac{1}{2}\right)^n.$$

Hence

$\forall_{\varepsilon>0}\ \exists_{\bar n, \tilde q}\ \forall_{q>\tilde q}$ $$d\left(u(k_{l_{m\dots q}})-u\right) \le 2\cdot\max_{n=1,\dots,\bar n}\overbrace{|u_n(k_{l\dots q})-u_n|}^{<\frac{\varepsilon}{4}} + \overbrace{\left(\frac{1}{2}\right)^{\bar n-1}}^{<\frac{\varepsilon}{2}} < \varepsilon.$$

We have constructed a convergent subsequence.

Hence (E,d) is (sequentially) compact. □

Unit ball in l_∞?

$$\|u\| = \max_n \{|u_n| : n \in \mathbb{N}\}$$

$$B_1(0) = \{u \in l_\infty : \|u\| \leq 1\} = \{u \in l_\infty : \max_{n\in\mathbb{N}} |u_n| \leq 1\} =$$

$$= \{(u_1, u_2, \dots) : \forall i \, |u_i| \leq 1\}.$$

Unit ball in (E, d)?

$$B_1(0) = \{u \in E : d(u,0) \leq 1\} = \{u \in E : \sum_{n\geq 0} \frac{\min\{1, |u_n|\}}{2^n} \leq 1\}$$

E.g. $(1, 1, 0, 0, \dots) \notin B_1(0)$ because $1 + \frac{1}{2} = \frac{3}{2} > 1$

E.g. $(0, 0, 7, 1567, (1567)^2, \dots) \in B_1(0)$ because $\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2} \leq 1$

In finite dimensional spaces, $B_1(0)$ is bounded & closed and thus compact.

In infinite dimensional normed spaces,

Def. $B_1(0) = \{x \in X : \|x\| \leq 1\}$. Norm: $\|x\| = 0 \iff x = 0$, $\|\alpha x\| = |\alpha| \cdot \|x\|$, Triangle inequality

1° Since we are dealing with a vector space, there exists a basis $\mathcal{B} = \{x_i\}_{i\in I}$ such that $\forall x \in X \; \exists \{\alpha_i\}_{i\in I} \; x = \sum_{i\in I} \alpha_i x_i$.

2° Infinite dimensional $\iff$ the basis is not a finite set.

3° Take a sequence of $(x_i)_{i\in J}$, J - countable subset of I.

4° For all $i \in J$, $\|x_i\| > 0$ (by 1st axiom). Moreover, there exists $\alpha_i > 0$ such that $\|\alpha_i x_i\| = \alpha_i \|x_i\| = 1$ (by 2nd axiom).

5° So we have an infinite sequence $(a_i x_i)_{i\in J} \subset B_1(0)$.

6° This sequence does not have a convergent subsequence because $\{x_i\}_{i\in J} \subset \mathcal{B}$, they don't repeat and $\|\alpha_i x_i\|$ is bounded away from zero $\square$

<u>Theorem</u> $(X,d), (Y,\rho)$ – metric spaces

$f: X \to Y$ continuous

If C – compact in X then $f(C)$ – compact in Y.

<u>Proof</u> Let $\{y_n\} \subset f(C)$, take $\{x_n\} \subset C$ such that $f(x_n) = y_n$.

By seq. compactness of C, there exists $\{x_{n_k}\} \to x \in C$.

Then $\lim\limits_{k\to\infty} y_{n_k} = \lim\limits_{k\to\infty} f(x_{n_k}) = f\left(\lim\limits_{k\to\infty} x_{n_k}\right) = f(x) \in f(C)$.

Hence $\{y_n\}$ has a convergent subsequence $\Rightarrow$ seq. compactness of $f(C)$ $\square$

<u>Weierstrass theorem.</u>

C – compact, $f: C \to \mathbb{R}$ continuous. Then f – bounded on C and attains both its min & max on C.

There exist $x_m, x_M \in C$ such that $f(x_M) = \sup f(C)$, $f(x_m) = \inf f(C)$

<u>Proof.</u> $f(C)$ – compact, $f(C) \subset \mathbb{R} \Rightarrow$ closed & bounded.

<u>Theorem</u> $f: X \to Y$ continuous, $(X,d), (Y,\rho)$ – metric spaces, $C \subset X$ – compact

Then f – <u>uniformly</u> continuous.

<u>Theorem</u> f – locally Lipschitz on A – compact, then Lipschitz on it.

TUTORIAL 18, EXERCISE 2

$x_0 \geq 0$ given

$$\max_{\forall t\geq 1 \quad x_t \in [0, f(x_{t-1})]} \sum_{t\geq 1} \beta^t u(\overbrace{f(x_{t-1}) - x_t}^{c_t}) \qquad c_t + x_t = f(x_{t-1})$$

- $f: \mathbb{R}_+ \to \mathbb{R}_+$, bounded, continuous
  $u: \mathbb{R}_+ \to \mathbb{R}_+$ — " —

$$E = \{(x_t)_{t\in\mathbb{N}}\} \text{ with } d(x,y) = \sum_{n\geq 0} \frac{\min\{1, |x_n - y_n|\}}{2^n}$$

[ONLY THE BEGINNING MATTERS]

a) $V((x_t)_{t\in\mathbb{N}})$, $V: E \to \mathbb{R}$

(E, d) compact !! Let $V(x) = \sum_{t\geq 1} \beta^t u(f(x_{t-1}) - x_t)$

Let $d(x,y) < \delta$. ~~We get~~ This means that for all $n \in \mathbb{N}$, $|x_n - y_n| < \tilde{\delta}$. (particular property of (E,d))

Hence $|V((x_t)) - V((y_t))| = |\sum_{t\geq 1} \beta^t (u(f(x_{t-1}) - x_t) - u(f(y_{t-1}) - y_t)|$

By continuity of f, u,
$|f(x_{t-1}) - f(y_{t-1})| < \frac{\varepsilon}{7}$, etc.,
$|V(x_t) - V(y_t)| \leq |\sum_{t\geq 1}^{\infty} \beta^t \frac{\varepsilon}{7}| = \frac{\varepsilon}{7} \cdot \frac{\beta}{1-\beta}$. V-continuous.

b) $\forall t \; (0 \leq x_t \leq f(x_{t-1}))$

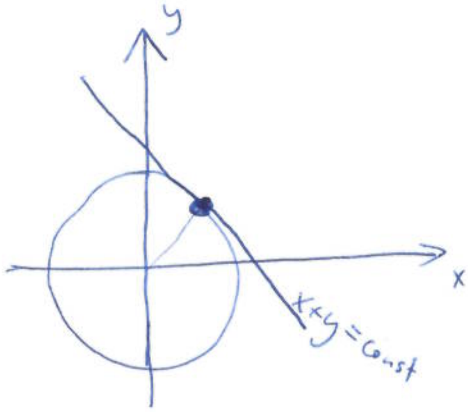
The set is: $\begin{cases} 0 \leq x_1 \leq f(x_0) \\ 0 \leq x_2 \leq f(x_1) \text{ etc.} \end{cases}$
$A \subset E$

- Any intersection of closed sets is closed. So A is closed.
- $A \subset E$ and A is a closed subset of a compact space, so it's compact
- It is also bounded : $E \subset B_{25}(0)$ ...

c) WEIERSTRASS

Another example why infinite horizon matters. (A9)

Let n - finite. $||v|| = \left(\sum_{t=1}^{n} v_t^2\right)^{\frac{1}{2}}$ (Euclidean norm).



It is easy to show for any $n \in \mathbb{N}$
that $\sum_{t=1}^{n} v_t$ is maximized on the ball $B_r(0)$ for $\underline{v_1 = \ldots = v_t}$.

$$B_r(0) = \{ v \in \mathbb{R}^n : ||v|| \leq r \} = \{ v \in \mathbb{R}^n : \sum_{t=1}^{n} v_t^2 \leq r^2 \}.$$

IN FACT, EQUALITY WILL HOLD IN THE MAX

Lagrangean reads:

$$\mathcal{L}(v_1, \ldots, v_n) = \sum_{t=1}^{n} v_t + \lambda \left(\sum_{t=1}^{n} v_t^2 - r^2 \right)$$

$$\forall_{t=1,\ldots,n} \quad \frac{\partial \mathcal{L}}{\partial v_t} = 1 + \lambda \cdot 2 v_t = 0 \Rightarrow \lambda = -\frac{1}{2 v_t} \quad (v_t \neq 0)$$

Hence $\frac{1}{2v_t} = \frac{1}{2v_s}$ $\forall s \neq t$ and thus $\underbrace{v_t = v_s}_{= \bar{v}}$ $\forall s \neq t$.

Using the boundary condition, we obtain:

$$\sum_{t=1}^{n} v_t^2 = \sum_{t=1}^{n} \bar{v}^2 = n \bar{v}^2 = r^2 \Rightarrow \underline{\bar{v}^2 = \frac{r^2}{n}}.$$

For the $\underline{\text{max}}$, take positive values: $\bar{v} = \frac{r}{\sqrt{n}}$.

The maximum value attained at $\left(\frac{r}{\sqrt{n}}, \ldots, \frac{r}{\sqrt{n}}\right)$ is

$$\sum_{t=1}^{n} \frac{r}{\sqrt{n}} = n \cdot \frac{r}{\sqrt{n}} = \underline{\sqrt{n} \cdot r}.$$

(check SOCs...)

Clearly, as $n \to \infty$, the maximum value $\varphi_n = r \cdot \sqrt{n} \xrightarrow{n \to \infty} \infty$.

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Let us show that

$$\max \sum_{t=1}^{\infty} v_t \quad \text{s.t.} \left(\sum_{t=1}^{\infty} v_t^2\right)^{\frac{1}{2}} \leq r$$

$B_r(0) = \{v \in V : \|v\| \leq r\}$.

does not exist.

- We will construct a sequence of sequences v, divergent to $+\infty$ wrt. the objective fct.

$v \in B_r(0)$

- Let

$$V_{(1)} = (r, 0, 0, 0, \ldots), \quad \|v_{(1)}\| = r, \quad \sum_{t=1}^{\infty} v_t = r.$$

$$V_{(2)} = \left(\frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}}, 0, 0, \ldots\right), \quad \|v_{(2)}\| = \left(\frac{r^2}{2} + \frac{r^2}{2}\right)^{\frac{1}{2}} = r, \quad \sum_{t=1}^{\infty} v_t = \frac{2}{\sqrt{2}} r = \sqrt{2} \cdot r$$

$$V_{(3)} = \left(\frac{r}{\sqrt{3}}, \frac{r}{\sqrt{3}}, \frac{r}{\sqrt{3}}, 0, \ldots\right), \quad \|v_{(3)}\| = \left(\frac{r^2}{3} + \frac{r^2}{3} + \frac{r^2}{3}\right)^{\frac{1}{2}} = r, \quad \sum v_t = \sqrt{3} \cdot r$$

$\vdots$

for $v_{(n)}$, $\|v_{(n)}\| = r$ but $\sum_{t=1}^{\infty} v_t = r \cdot \sqrt{n}$.

Hence, the objective function is <u>not</u> bounded from above.
The maximum does not exist.

- Two associated results:

— COORDINATE-WISE CONVERGENCE.

It is easy to show that $\forall t \geq 1$ $\forall n \geq t$ $v_{(n)t} = \frac{r}{\sqrt{n}}$.

Hence $\underbrace{\lim_{n \to \infty} v_{(n)t} = 0 \quad \forall t \geq 1}$, even though $\|v_{(n)}\| = r$ $\forall n \in \mathbb{N}$.

— NORM IS WELL-DEFINED:

- $\|v\| = 0 \Leftrightarrow \left(\sum_{t=1}^{\infty} v_t^2\right)^{\frac{1}{2}} = 0 \Leftrightarrow \sum_{t=1}^{\infty} v_t^2 = 0 \Leftrightarrow \forall t \geq 1 \; v_t = 0 \Leftrightarrow v = 0.$
- $\|\alpha v\| = \left(\sum_{t=1}^{\infty} \alpha^2 v_t^2\right)^{\frac{1}{2}} = \left(\alpha^2 \sum_{t=1}^{\infty} v_t^2\right)^{\frac{1}{2}} = |\alpha| \left(\sum_{t=1}^{\infty} v_t^2\right)^{\frac{1}{2}} = |\alpha| \cdot \|v\|.$
- $\|v + w\| = \left(\sum_{t=1}^{\infty} (v_t + w_t)^2\right)^{\frac{1}{2}} \leq \left(\sum_{t=1}^{\infty} v_t^2\right)^{\frac{1}{2}} + \left(\sum_{t=1}^{\infty} w_t^2\right)^{\frac{1}{2}} = \|v\| + \|w\|.$

MINKOWSKI INEQUALITY